TRINITY COLLEGE

Year 12 Mathematics Specialist Units 3, 4
Test 1 2020

Section 1 Calculator Free

Complex Numbers and Functions

STUDENT'S NAME SOLUTIONS (PRESSER)

DATE: Wednesday 4 March **TIME:** 28 minutes **MARKS:** 29

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (6 marks)

(a) Express $5\,cis\dfrac{5\pi}{6}$ in the form $z = a + bi$ [3]

$$= 5\left[\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right]$$

$$= -\frac{5\sqrt{3}}{2} + i \times 5 \times \frac{1}{2}$$

$$= -\frac{5\sqrt{3}}{2} + \frac{5}{2}i$$

✓ expands cis

✓ exact values

✓ answer

(b) Express $\dfrac{\overline{2-i}}{(1+i)^2}$ in the form $z = a + bi$ [3]

$$= \frac{2+i}{2i} \times \frac{i}{i}$$

$$(1+i) = \sqrt{2}\,cis\frac{\pi}{4}$$

$$(1+i)^2 = 2\,cis\frac{2\pi}{4}$$

$$= 2i$$

$$= \frac{2i-1}{-2}$$

$$= \frac{1}{2} - i$$

✓ conjugate

✓ $(1+i)^2$

✓ answer

2. (5 marks)

Solve $z^4 + 8i = 0$. Answers may be given in polar form.

$$\Rightarrow \quad z^4 = -8i$$

$$\Rightarrow \quad z^4 = 8 \operatorname{cis}\left(-\frac{\pi}{2}\right)$$

✓ polar

$$\Rightarrow \quad z = \left[8 \operatorname{cis}\left(-\frac{\pi}{2} + 2k\pi\right)\right]^{1/4}$$

$$\Rightarrow \quad z_0 = 8^{1/4} \operatorname{cis}\left(-\frac{\pi}{8}\right)$$

$$\text{angle} = \frac{2\pi}{4} = \frac{4\pi}{8}$$

$$z_1 = 8^{1/4} \operatorname{cis} \frac{3\pi}{8}$$

$$z_2 = 8^{1/4} \operatorname{cis} \frac{7\pi}{8}$$

$$z_3 = 8^{1/4} \operatorname{cis} \frac{11\pi}{8} = 8^{1/4} \operatorname{cis}\left(-\frac{5\pi}{8}\right)$$

✓ z_0

✓ angle separation

✓ z_1 + z_2

✓ z_3

3. (7 marks)

Consider the expression $z^4+3z^3-3z^2+3z-4$

(a) Show that $z-i$ is a factor of the above expression. [2]

$z-i$ is a factor $\Rightarrow$ $z=i$ is a root

$\therefore$ $(i)^4+3(i)^3-3(i)^2+3(i)-4$ ✓ uses i

$= 1 - 3i + 3 + 3i - 4$ ✓ substitutes & expands

$= 0$ $\therefore$ $z-i$ is a factor

(b) State another factor for the above expression. [1]

$z+i$ ✓ answer

(c) Hence, or otherwise, solve $z^4+3z^3-3z^2+3z-4=0$ [4]

$(z-i)(z+i)(az^2+bz+c) = z^4+3z^3-3z^2+3z-4$

$(z^2+1)(az^2+bz+c) = z^4+3z^3-3z^2+3z-4$

✓ quadratic

$\Rightarrow$ $a=1$

$b=3$ (by inspection) ✓ coefficients

$c=-4$

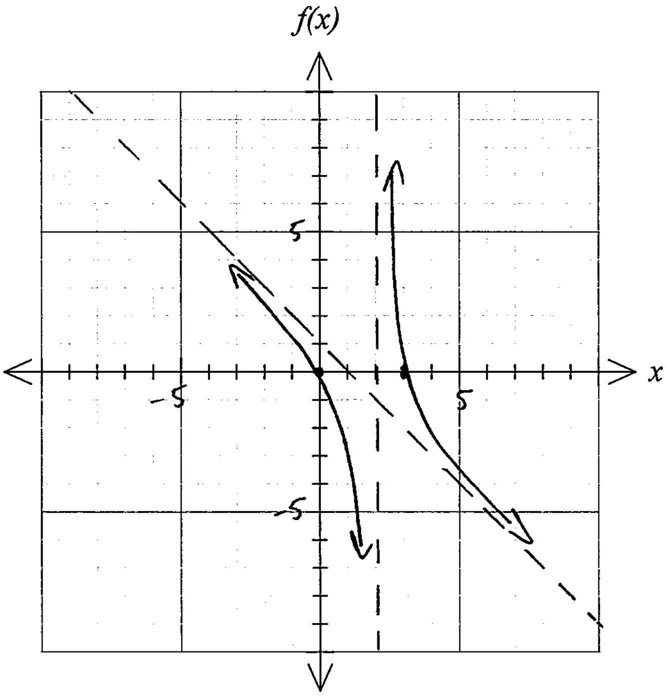
$\therefore$ $(z+i)(z-i)(z^2+3z-4)=0$

$(z+i)(z-i)(z+4)(z-1)=0$ ✓ factorised

$\therefore$ $z=\pm i, -4, 1$ ✓ solutions

4. (5 marks)

Sketch the function $f(x) = \dfrac{3x - x^2}{x-2}$, showing all intercepts, poles and asymptotes. It is not necessary to identify any stationary points.



✓ scale

✓ $x=3$ & shape

✓ pole

✓ oblique asymptote

✓ $x=0$ & shape

$$\begin{array}{r} -x + 1 \\ x-2 \overline{)\, -x^2 + 3x + 0} \\ \underline{-x^2 + 2x} \\ x + 0 \\ \underline{x - 2} \\ 2 \end{array}$$

y-int $f(0) = \dfrac{0}{-2}$

$= 0$

roots $0 = 3x - x^2$

$= x(3-x)$

$\therefore x = 0, 3$

$\therefore f(x) = \dfrac{2}{x-2} + (-x+1)$

$\therefore$ pole $x = 2$

oblique asymptote $y = -x + 1$

5. (6 marks)

Given $f(x) = \dfrac{3}{x^2 - 3}$ and $g(x) = \sqrt{x^2 - 1}$

(a) By considering the restricted domain $\{x : x \in \mathbb{R}, x \geq 0, x \neq \sqrt{3}\}$, determine $f^{-1}(x)$ and state the restricted range of $f^{-1}(x)$. [3]

let $y = \dfrac{3}{x^2 - 3}$

$\Rightarrow \quad yx^2 - 3y = 3$

$\Rightarrow \quad x^2 = \dfrac{3y + 3}{y}$

$\therefore \quad f^{-1}(x) = \sqrt{\dfrac{3x+3}{x}}$ (only +√ due to domain)

Range $f^{-1}(x) = \{y : y \in \mathbb{R}, y \geq 0, y \neq \sqrt{3}\}$

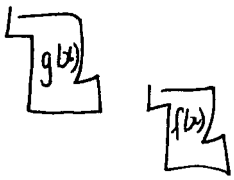
✓ rearranges
✓ $f^{-1}(x)$ in terms of x
✓ range

(b) Determine an expression for $f \circ g(x)$ and state the domain of $f \circ g(x)$. [3]

$f(\sqrt{x^2 - 1})$

$= \dfrac{3}{(\sqrt{x^2-1})^2 - 3}$

$= \dfrac{3}{x^2 - 4}$



✓ expression
✓ $x \neq \pm 2$
✓ $x \leq -1, x \geq 1$

from $\sqrt{x^2-1}$ $\quad x^2 - 1 \geq 0$ 

$\Rightarrow x \geq 1, \ x \leq -1$

from $x^2 - 4$ $\quad x^2 - 4 \neq 0$

$x \neq \pm 2$

$\therefore$ Domain $f \circ g(x) = \{x : x \in \mathbb{R}, x \leq -1, x \geq 1, x \neq \pm 2\}$

TRINITY COLLEGE

Year 12 Mathematics Specialist Units 3, 4
Test 1 2020

Section 2 Calculator Assumed
Complex Numbers and Functions

STUDENT'S NAME ____________________

DATE: Wednesday 4 March **TIME:** 22 minutes **MARKS:** 22

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Special Items: Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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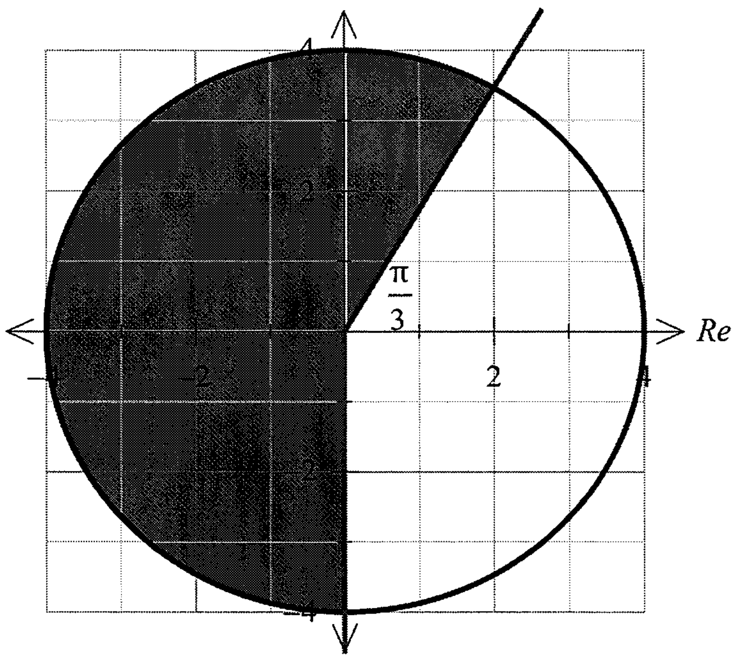
6. (11 marks)

(a) Describe fully the shaded region below [3]



$|z| \leq 4$ ✓

$Arg(z) \geq \frac{\pi}{3}$ ✓

$Arg(z) \leq -\frac{\pi}{2}$ ✓

$\{z : z \in \mathbb{C},\ |z| \leq 4,\ Arg(z) \geq \frac{\pi}{3},\ Arg(z) \leq -\frac{\pi}{2}\}$

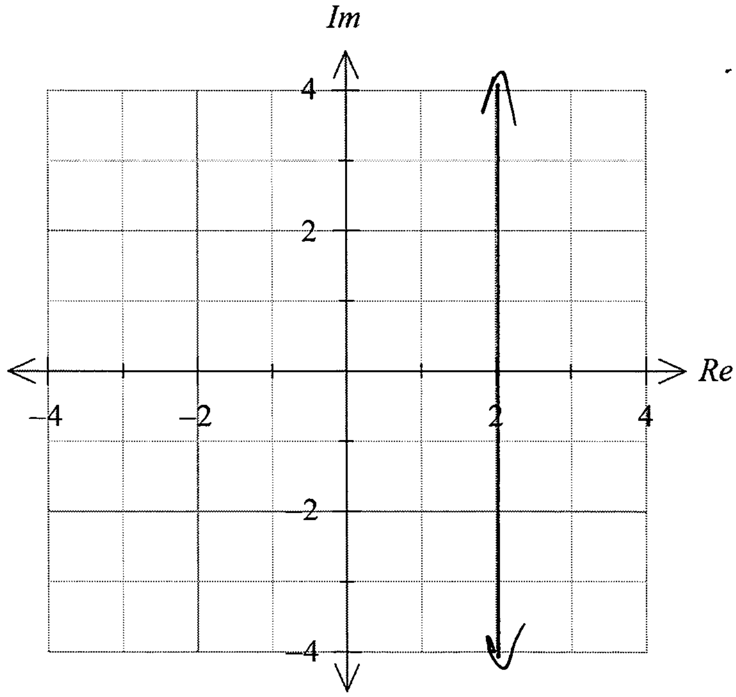
(b) Sketch the set of locus $\{z : z + \bar{z} = 4\}$ [3]

let $z = x + iy$

$\therefore x + iy + x - iy = 4$

$2x = 4$

$x = 2$

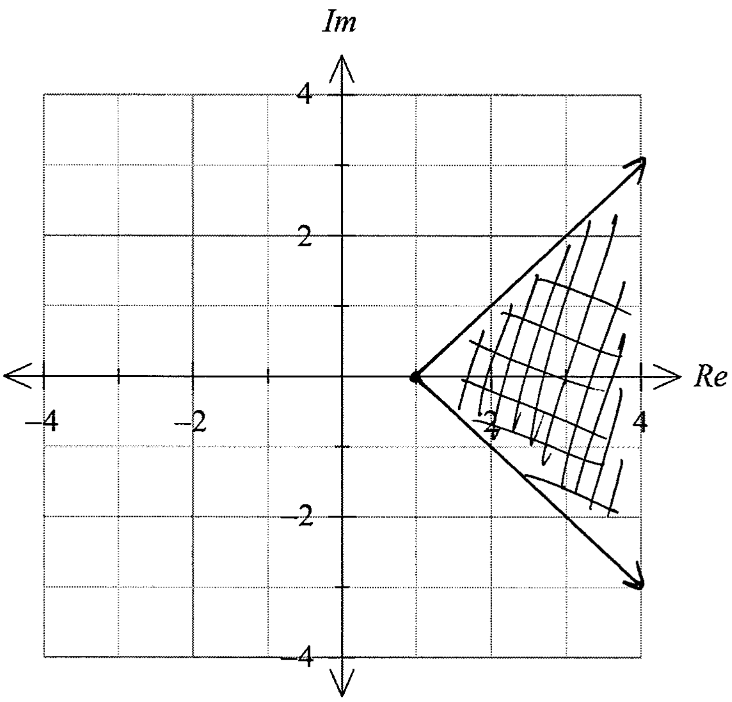


✓ let $z = x+iy$

✓ $x = 2$

✓ Argand diagram

(c) Sketch the set of locus $\left\{z : |Arg(z-1)| \leq \frac{\pi}{4}\right\}$ [3]



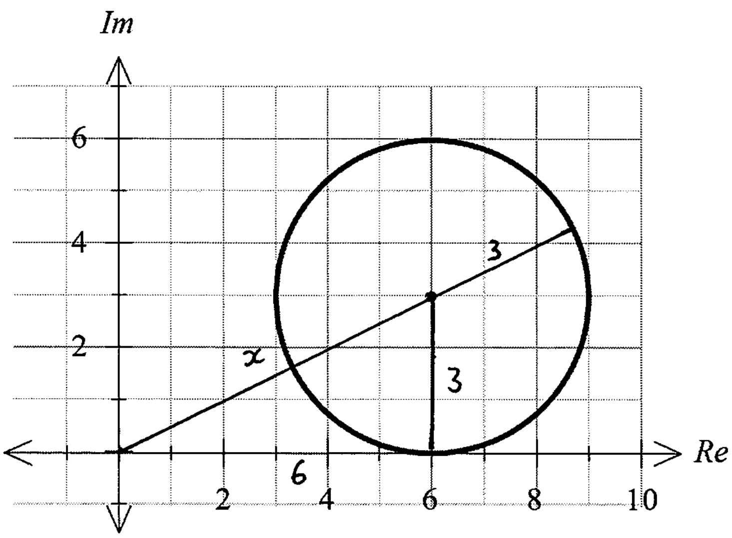
✓ vertex at $Re(z) = 1$

✓ Ray $\frac{\pi}{4}$ and shaded

✓ Ray $-\frac{\pi}{4}$ and shaded

(d) The sketch of the locus of a complex number $\{z : |z-6-3i| = 3\}$ is given below:

$z-(6+3i)$



distance from 0

Determine the maximum value for $|z|$ as an exact value. [2]

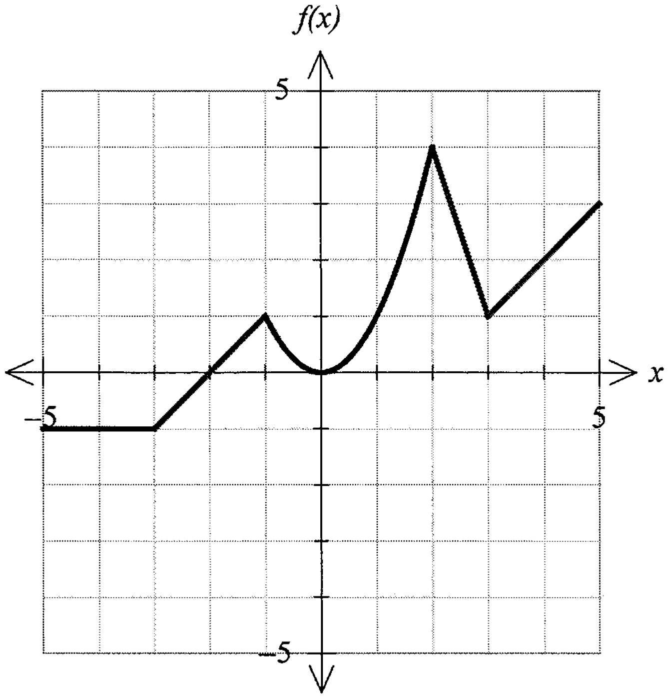
$$\begin{aligned} \max |z| &= x + 3 \\ &= \sqrt{6^2 + 3^2} + 3 \\ &= \sqrt{45} + 3 \\ &= 3\sqrt{5} + 3 \end{aligned}$$

✓ $\sqrt{45}$

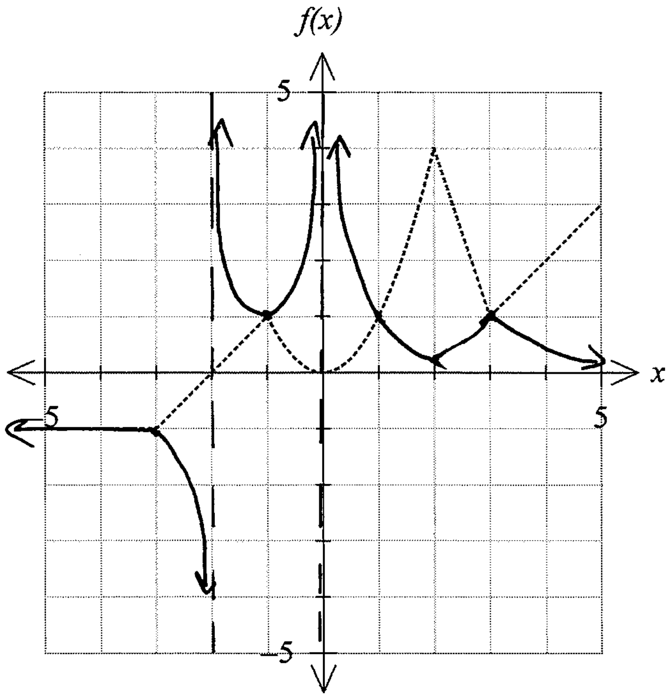
✓ $\sqrt{45} + 3$

7. (6 marks)

Consider the following function



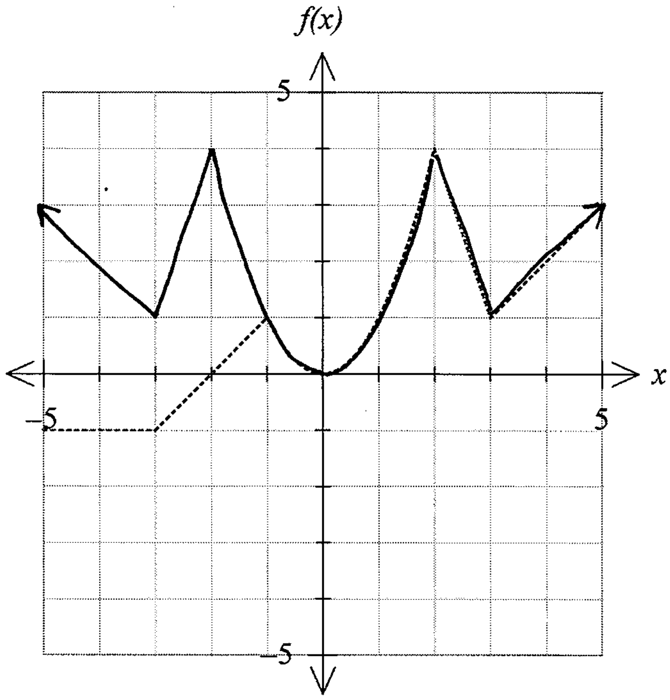
(a) Sketch $\dfrac{1}{f(x)}$ [2]



✓ $x \leq 0$

✓ $x > 0$

(c) Sketch $|f(|x|)|$ [2]



✓ $x \geq 0$

✓ $x < 0$

(d) Hence, or otherwise, solve $f(x)|f(|x|)| = 1$ for $x \geq 0$ [2]

$\Rightarrow |f|x|| = \frac{1}{f(x)}$

Where the two graphs intersect

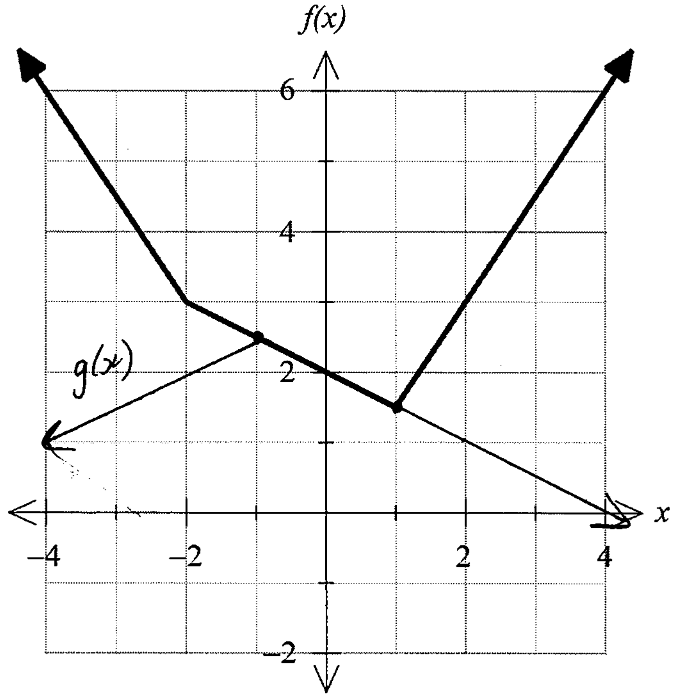
$\Rightarrow x = 1, 3$

✓ rearrange

✓ sol/ns

8. (5 marks)

The graph of $f(x) = |x-1| + \left|\frac{x}{2}+1\right|$ is given below:



✓ vertex $(-1, \frac{5}{2})$

✓ inverted

The solution to the equation $a|x+b|+c = |x-1| + \left|\frac{x}{2}+1\right|$ is $\{x : -1 \le x \le 1\}$.

(a) Sketch a possible graph of $g(x) = a|x+b|+c$ on the axes above. [2]

(b) Determine the values of the real constants a , b and c . [3]

$a = -\frac{1}{2}$ (think gradient) ✓

$b = 1$ (think vertex) ✓

$c = \frac{5}{2}$ (think tp .) ✓